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ON COMPLETELY REDUCIBLE SOLVABLE SUBGROUPS OF GL(n, Δ)

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ABSTRACT

Let Δ be a finite field and denote by GL(n,Δ) the group of $n\times n$ nonsingular matrices defined over Δ. Let $R\subseteq \mathrm{GL}(n,\Delta)$ be a solvable, completely reducible subgroup of maximal order. For $|\Delta|\geqq 2$, $|\Delta|\neq 3$ we give bounds for $|R|$ which improve previous ones. Moreover for $|\Delta|=3$ or $|\Delta|>13$ we determine the structure of R, in particular we show that R is unique, up to conjugacy.

0. Introduction

Let Δ be a finite field of order q. Denote by GL(n,Δ) the group of $n\times n$ nonsingular matrices defined over Δ. In this paper we are concerned with solvable, completely reducible subgroups of GL(n,Δ). Denote by $b(n,q)$ the maximal possible order of a solvable, completely reducible subgroup of GL(n,Δ). T. R. Wolf ([5], thm. 3, p. 1108) showed that $b(n,q)\leqq q^{(1+\beta)n}/24^{1/3}$ where $6/5<\beta<5/4$, for all $q\geqq 2$ and $n\geqq 1$. We shall improve this bound for all $q\geqq 2$, $q\neq 3$ and $n\geqq 1$. Moreover, we shall determine, up to conjugacy, the structure of a solvable, completely reducible subgroup of GL(n,Δ) of order $b(n,q)$, for $q>13$ and $n\geqq 1$. In fact, we shall show the following: Let S_n be the symmetric group on n letters. Denote by γ_n the maximal possible order of a solvable subgroup of S_n. Then we have the following theorem:

THEOREM A.[††] *Denote by $b(n,q)$ the maximal possible order of a solvable, completely reducible subgroup of* GL(n,Δ), *where Δ is a finite field of order q and $q\geqq 11$. Then*

$$b(n,q)\leqq \gamma_n q^n \qquad \text{for all } n\geqq 1,$$

[†]This work is part of a Ph.D. thesis done at the Hebrew University under the supervision of Professor A. Mann.

[††] The assertion of Theorem A is in fact true for $q\geqq 8$ also. The proof for that can be found in [3]. See also Section 4 at the end of the paper.

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Dixon ([1], thm. 3, p. 418) showed that $\gamma_n \leqq a^{n-1}$ where $a = 24^{1/3}$. If we set $\beta(q) = \log_q a$ for all $q \geqq 11$, then clearly $q^{\beta(q)} = a$. Hence, we have the following Corollary:

COROLLARY. *Let Δ and $b(n,q)$ be as in Theorem A. Then*:

$$b(n,q) \leqq q^{(1+\beta(q))n}/24^{1/3} \qquad \textit{for all } n \geqq 1. \tag{0.1}$$

Since $0 < \beta(q) < \frac{1}{2}$ and $\lim_{q\to\infty} \beta(q) = 0$, the bound given in (0.1) is indeed an improvement of the bound given in [5] for all $q \geqq 11$ and $n \geqq 1$.

We shall also prove the following theorem:

THEOREM B. *Let Δ be a finite field of order q, where $q > 13$ and let Δ^* be the multiplicative group of Δ. Let $R \subset \mathrm{GL}(n,\Delta)$ be a solvable, completely reducible subgroup of order $b(n,q)$, where $n \geqq 1$. Then there exists a solvable subgroup $\Gamma_n \subseteq S_n$ of maximal order, a solvable subgroup $\Gamma_{n-2} \subset S_{n-2}$ of maximal order, and a solvable irreducible subgroup $H \subseteq \mathrm{GL}(2,\Delta)$ of maximal order such that*:

(α) *If $n \not\equiv 2 \pmod 4$ or $n \equiv 6 \pmod{16}$, then R is conjugate in $\mathrm{GL}(n,\Delta)$ to the wreath product $\Delta^* \int \Gamma_n$.*

(β) *If $n \equiv 2 \pmod 4$ but $n \not\equiv 6 \pmod{16}$, then R is conjugate in $\mathrm{GL}(n,\Delta)$ to the direct product $H \times (\Delta^* \int \Gamma_{n-2})$.*

A. Mann ([2]) showed that all solvable subgroups of maximal order in S_n form a conjugacy class of subgroups in S_n. D. A. Suprunenko ([4], thm. 6, p. 167) determined up to conjugacy all maximal, irreducible, solvable subgroups of $\mathrm{GL}(2,\Delta)$. Hence, as a corollary we shall show:

COROLLARY. *If Δ is a finite field of order q and $q > 13$, then the set of all solvable, completely reducible subgroups of $\mathrm{GL}(n,\Delta)$, of order $b(n,q)$, form a conjugacy class of subgroups in $\mathrm{GL}(n,\Delta)$, for all $n \geqq 1$.*

1. Preliminaries and notations

Throughout this paper the following notations will be used:

(a) Δ shall denote a finite field and q shall denote its order. We denote by Δ^* the multiplicative group of Δ.

(b) We shall fix a solvable subgroup of maximal order in S_n and denote it by Γ_n. We denote the order of Γ_n by γ_n.

(c) Let $\Gamma^{(1)} \subset S_{n_1}, \Gamma^{(2)} \subset S_{n_2}, \ldots, \Gamma^{(t)} \subset S_{n_t}$ be subgroups. Set $n = \sum_{i=1}^{t} n_i$. By the direct product $\Gamma^{(1)} \times \Gamma^{(2)} \times \cdots \times \Gamma^{(t)}$ we mean the subgroup of S_n acting on the set of indices $\{1,2,\ldots,n_1\}$ like $\Gamma^{(1)}$ does,..., and acting on the set indices

$\{\sum_{i=1}^{k-1} n_i + 1, \sum_{i=1}^{k-1} n_i + 2, \ldots, \sum_{i=1}^{k} n_i\}$ like $\Gamma^{(k)}$ does on $\{1, \ldots, n_k\}$ respectively, where $2 \leqq k \leqq t$.

(d) Let $H_i \subseteq \mathrm{GL}(n_i, \Delta)$ be subgroups for $1 \leqq i \leqq t$. By the direct product $H_1 \times H_2 \times \cdots \times H_t$ we mean the subgroup of $\mathrm{GL}(n, \Delta)$ consisting of all matrices of the form $\mathrm{diag}(h_1, h_2, \ldots, h_t)$ where $h_i \in H_i$ for $1 \leqq i \leqq t$ and $n = \sum_{i=1}^{t} n_i$.

(e) The maximal order of a solvable, completely reducible subgroup of $\mathrm{GL}(n, \Delta)$ shall be denoted by $b(n, q)$.

(f) Let G be a group and let $G_1, G_2 \subset G$ be subgroups. We shall write $G_1 \sim_G G_2$ to denote that G_1 and G_2 are conjugate subgroups in G. If the group G is understood from the context we shall merely write $G_1 \sim G_2$.

From now on we shall assume $q \geqq 11$ and we mention that all numerical variables in this paper shall be integer variables.

This paper extensively uses the following concepts:

(1) The wreath product of permutation groups (see [4], p. 11). If $\Gamma' \subset S_n$ and $\Gamma'' \subset S_t$ are subgroups, then from now on $\Gamma' \int \Gamma''$ shall denote the wreath product of Γ' and Γ'' which is a subgroup of $S_{n \cdot t}$.

(2) The wreath product of a linear group and a permutation group (see [4], p. 106). If $R \subset \mathrm{GL}(n, \Delta)$ is a linear group and $\Gamma \subset S_t$ is a permutation group then from now on $R \int \Gamma$ shall denote the wreath product of R and Γ which is a subgroup of $\mathrm{GL}(n \cdot t, \Delta)$.

(3) Primitivity and imprimitivity of linear groups (see [4], chapter 15, p. 103).

The proof of Theorem A and Theorem B relies on results of D. A. Suprunenko [4] and A. Mann [2] which we state below.

(1.1) ([4], p. 139). Let M be a maximal primitive solvable subgroup of $\mathrm{GL}(n, \Delta)$, then M contains the following invariant series:

$$(E_n) \subset F \subset A \subset V \subset M$$

where we fix the following notations:

(1) M is a maximal primitive solvable subgroup of $\mathrm{GL}(n, \Delta)$.

(2) E_n is the identity element in $\mathrm{GL}(n, \Delta)$.

(3) F is a maximal abelian normal subgroup of M.

(4) V is the centralizer of F in M.

(5) A/F is maximal among the subgroups of M/F satisfying:

(i) A/F is an abelian normal subgroup of M/F.

(ii) $A/F \subseteq V/F$.

(1.2) ([4], lemma 1, p. 129) $F = K^*$, where K^* is the multiplicative group of a field extension K of $E_n \cdot \Delta$ and $[K : E_n \cdot \Delta]$ divides n. We fix the letter m to denote $m = [K : E_n \cdot \Delta]$ and we fix the letter r (like in [4]) to denote $r = n/m$.

(1.3) ([4], thm. 3, p. 141) $[A:F]=r^2$.

(1.4) ([4], thm. 15, p. 151) If $r=p_1^{l_1}p_2^{l_2}\cdots p_k^{l_k}$ is a prime factorization of r (where $p_i\neq p_j$ for $i\neq j$, $1\leqq i,j\leqq k$). Then V/A is isomorphic to a solvable subgroup of the direct product of k symplectic groups $\mathrm{Sp}(2l_j,p_j)$, $j=1,\ldots,k$.

(1.5) ([4], thm. 1, p. 138) Since V centralizes F, V can be viewed as a subgroup of $\mathrm{GL}(r,K)$ (where K is like in (1.2)) and we have: V is an absolutely irreducible subgroup of $\mathrm{GL}(r,K)$.

(1.6) ([4], Cor. 1, p. 130) $[M:V]\leqq m$.

(1.7) ([4], thm. 6, p. 167) Every maximal irreducible solvable subgroup of $\mathrm{GL}(2,\Delta)$ is conjugate in $\mathrm{GL}(2,\Delta)$ to one of the following three subgroups:
$G_{2,1}$ of order $2(q-1)^2$,
$G_{2,2}$ of order $2(q^2-1)$,
$G_{2,3}$ of order $24(q-1)$,
where $G_{2,1}$, $G_{2,2}$ and $G_{2,3}$ are as defined in ([4], chapter 19) and $G_{2,3}$ exists iff 2 does not divide q. We fix $G_{2,2}$ to denote the subgroup mentioned above.

(1.8) ([2]) Let $n\geqq 1$ be an integer. For convenience of notations denote by Γ_0 the empty set, and for a permutation group Γ, $\Gamma\int\Gamma_0=\Gamma_0$ and $\Gamma_0\times\Gamma=\Gamma$. Let $n=4k+t$ where $k\geqq 0$ and $0\leqq t\leqq 3$. Then Γ_n is conjugate in S_n to the following group:

(1) If $t=0$ then $\Gamma_n\sim S_4\int\Gamma_k$.

(2) If $t=1$ then:

(α) If $n\equiv 9 \pmod{16}$ but $n\not\equiv 25 \pmod{64}$ then Γ_n is conjugate in S_n to the direct product $(S_4\int\Gamma_{k-2})\times(S_3\int S_3)$.

(β) If $n\not\equiv 9 \pmod{16}$ or $n\equiv 25 \pmod{64}$ then $\Gamma_n\sim\Gamma_{n-1}\times S_1$.

(3) If $t=2$ then:

(α) If $n\equiv 6 \pmod{16}$ then Γ_n is conjugate in S_n to the direct product $(S_4\int\Gamma_{k-1})\times(S_3\int S_2)$.

(β) If $n\not\equiv 6 \pmod{16}$ then Γ_n is conjugate in S_n to the direct product $\Gamma_{n-2}\times S_2$.

(4) If $t=3$ then $\Gamma_n\sim\Gamma_{n-3}\times S_3$.

2. Preliminary lemmas

In order to prove Theorem A and Theorem B we shall need several lemmas, which we prove in this section.

(2.1) *Let $n\geqq 4$ and $n\neq 5$, then $\gamma_n>n^2$*

PROOF. For $n = 4$ we have $\gamma_4 = |S_4| = 24$ hence $\gamma_4 > 4^2$. For $n = 6, 7$ clearly S_n contains $S_3 \int S_2$ which is solvable of order 72, hence $\gamma_6 > 6^2$ and $\gamma_7 > 7^2$. For $n \geqq 8$ we proceed by induction on n. Assume by induction that for any integer n_1 such that $6 \leqq n_1 < n$ we have $\gamma_{n_1} > n_1^2$. Since clearly S_n contains the group $\Gamma_{n-2} \times S_2$, which is a solvable group of order 2 γ_{n-2}, then by induction we have:

$$\gamma_n \geqq 2\gamma_{n-2} > 2(n-2)^2 = 2n^2 - 8n + 8.$$

But it is easy to see that for all $n \geqq 8$ we have $2n^2 - 8n + 8 > n^2$ hence we conclude: $\gamma_n > n^2$. ∎

(2.2) *For all $n \geqq 2$ we have $\gamma_n > (1.4)^n$.*

PROOF. If n is even, set $n = 2k$. Then S_n contains the direct product

$$\underbrace{S_2 \times S_2 \times \cdots \times S_2}_{k \text{ times}},$$

which is a solvable group of order 2^k Hence we have $\gamma_n \geqq 2^k = 2^{n/2} > (1.4)^n$.

If n is odd, set $n = 2k + 1$. Then S_n contains the direct product

$$S_3 \times \underbrace{S_2 \times \cdots \times S_2}_{k-1 \text{ times}},$$

which is solvable of order $6 \cdot 2^{k-1}$. Hence we have

$$\gamma_n \geqq 6 \cdot 2^{k-1} = 3 \cdot 2^k > 2^{k+1} = 2^{n+1/2} > 2^{n/2} > (1.4)^n. \quad ∎$$

(2.3) *For any $n \geqq 3$ we have $\gamma_n(q-1)^n > nq^n$.*

PROOF. We remind the reader that we always assume $q \geqq 11$. For $3 \leqq n \leqq 9$ the assertion of (2.3) can easily be checked, noticing that $\gamma_5 \geqq 24$ (since $S_5 \supset S_4$), $\gamma_6 \geqq 72$ (since $S_6 \supset S_3 \int S_2$), $\gamma_7 \geqq 144$ (since $S_7 \supset S_4 \times S_3$), $\gamma_8 \geqq (24)^2 2$ (since $S_8 \supset S_4 \int S_2$) and $\gamma_9 \geqq 6^4$ (since $S_9 \supset S_3 \int S_3$).

For $n \geqq 10$, since $q \geqq 11$ we have $q/(q-1) \leqq 1.1$. By (2.2) we have $\gamma_n > (1.4)^n$. Hence the assertion of (2.3) follows from the inequality $(1.4)^n > n(1.1)^n$, which can easily be shown by induction. ∎

(2.4) *Let $n \geqq 2$. Assume that Γ_n is a transitive subgroup of S_n. Then we have $\gamma_{2n}/\gamma_n \geqq 12^{n/3}$.*

PROOF. By (1.8) since Γ_n is transitive we must have either $n = 2, 3, 6, 9$ or $n = 4k$ where $k \geqq 1$. For $n = 2, 3, 6, 9$ the assertion of (2.4) can easily be checked, using (1.8). For $n = 4k$ we have $2n = 8k$, hence S_{2n} contains the group $\Gamma_8 \int \Gamma_k$ which is clearly solvable of order $\gamma_8^k \cdot \gamma_k$. By (1.8) $\Gamma_n = S_4 \int \Gamma_k$. Hence we have $\gamma_{2n}/\gamma_n \geqq \gamma_8^k \cdot \gamma_k/\gamma_4^k \cdot \gamma_k = \gamma_8^k/\gamma_4^k$. But by (1.8) we have $\Gamma_8 = S_4 \int S_2$. Hence $\gamma_8 = (24)^2 \cdot 2$. Hence we have

$$\gamma_{2n}/\gamma_n \geqq [(24)^2 \cdot 2]^k/(24)^k = 48^k = 48^{n/4}.$$

Now since $48^{1/4} > 12^{1/3}$, we conclude $\gamma_{2n}/\gamma_n \geqq 12^{n/3}$. ∎

(2.5) *For any* $n \geqq 2$ *such that* $n \not\equiv 2 (\mathrm{mod}\ 4)$ *or* $n \equiv 6 (\mathrm{mod}\ 16)$ *we have* $\gamma_n \geqq 2\frac{2}{3}\gamma_{n-2}$.

PROOF. Set $n = 4k + t$, where $k \geqq 0$ and $0 \leqq t \leqq 3$. We distinguish 4 cases according as $t = 0, 1, 2, 3$ respectively.

Case 1. $t = 0$

In this case $n = 4k = 4(k-1)+4$. Since clearly $S_n \supset (S_4 \int \Gamma_{k-1}) \times S_4$ we have $\gamma_n \geqq |S_4 \int \Gamma_{k-1}| \cdot |S_4|$. Since $\gamma_{n-1} \geqq \gamma_{n-2}$ then by

(1.8) $$\gamma_{n-2} \leqq \gamma_{n-1} = |S_4 \int \Gamma_{k-1}| \cdot |S_3|.$$

Hence we have: $\gamma_n/\gamma_{n-2} \geqq |S_4|/|S_3| = 4 > 2\frac{2}{3}$.

Case 2. $t = 1$

In this case $n = 4k + 1 = 4(k-1)+5$ and $n - 2 = 4(k-1)+3$.

Clearly we have $\gamma_n \geqq |S_4 \int \Gamma_{k-1}| \cdot |S_4|$. By (1.8) we have $\gamma_{n-2} = |S_4 \int \Gamma_{k-1}| \cdot |S_3|$. Hence we have: $\gamma_n/\gamma_{n-2} = |S_4|/|S_3| = 4 > 2\frac{2}{3}$.

Case 3. $t = 2$

By the assumption on n we have $n \equiv 6 (\mathrm{mod}\ 16)$. Set $n = 4k + 2$, then we must have $k \equiv 1 (\mathrm{mod}\ 4)$. We distinguish 2 cases (α) and (β) as follows:

(α) $k \equiv 9 (\mathrm{mod}\ 16)$ but $k \not\equiv 25 (\mathrm{mod}\ 64)$

By (1.8) we have

$$\Gamma_{n-2} \sim S_4 \int \Gamma_k \sim S_4 \int (\Gamma_{k-9} \times \Gamma_9) \sim (S_4 \int \Gamma_{k-9}) \times (S_4 \int \Gamma_9) \sim (S_4 \int \Gamma_{k-9}) \times \Gamma_{36}.$$

Now since $n = 4(k-9)+38$ we clearly have $\gamma_n \geqq |S_4 \int \Gamma_{k-9}| \times |\Gamma_{38}|$. Hence by the above and by (1.8) we have

$$\gamma_n/\gamma_{n-2} \geqq \gamma_{38}/\gamma_{36} = |S_4 \int S_4 \int S_2| \cdot |\Gamma_6|/|S_4 \int \Gamma_9|$$
$$= (24)^{10} \cdot 2 \cdot 2^3 \cdot 3^2/(24)^9 \cdot 2^4 \cdot 3^4 = 24/9 = 2\tfrac{2}{3}$$

as asserted.

(β) $k \not\equiv 9 (\mathrm{mod}\ 16)$ or $k \equiv 25 (\mathrm{mod}\ 64)$

By (1.8) we must have $\Gamma_{n-2} \sim (S_4 \int \Gamma_{k-1}) \times S_4$. But again by (1.8) we have $\Gamma_n \sim (S_4 \int \Gamma_{k-1}) \times \Gamma_6$. Hence we have $\gamma_n/\gamma_{n-2} = |\Gamma_6|/|S_4| = 72/24 = 3 > 2\frac{2}{3}$.

Case 4. $t=3$

In this case $n=4k+3$. We distinguish 2 cases (α) and (β) as follows:

(α) $n-2\equiv 9 \pmod{16}$ but $n-2\not\equiv 25 \pmod{64}$

By (1.8) we have $\Gamma_{n-2}\sim(S_4\int\Gamma_{k-2})\times\Gamma_9$. But since $4k+3=4(k-2)+11$ we have $\gamma_n\geqq|S_4\int\Gamma_{k-2}|\times\Gamma_{11}$. Hence by (1.8) we have

$$\gamma_n/\gamma_{n-2}\geqq|\Gamma_{11}|/|\Gamma_9|=(24)^2\cdot 2\cdot 6/6^4=2\cdot 4^2\cdot 6^3/6^4=32/6>2\tfrac{2}{3}.$$

(β) $n-2\not\equiv 9 \pmod{16}$ or $n-2\equiv 25 \pmod{64}$

By (1.8) we have $\Gamma_{n-2}\sim S_4\int\Gamma_k$ and $\Gamma_n\sim(S_4\int\Gamma_k)\times S_3$. Hence we have: $\gamma_n/\gamma_{n-2}=|S_3|=6>2\tfrac{2}{3}$. ■

(2.6) *Let p be a prime and let $l\geqq 1$. Then we have*

(α) *For any $q\geqq 13$ the inequality $(q-1)^{p^l-1}>p^{2l^2+l}$ holds.*

(β) *For $q=11$ if $p^l\neq 4$ the inequality $(q-1)^{p^l-1}>p^{2l^2+l}$ holds.*

PROOF. The proof follows easily by induction. ■

(2.7) *Let p be a prime and let $l\geqq 1$. Let G be the symplectic group* $\mathrm{Sp}(2l,p)$. *Then we have $|G|<p^{2l^2+l}$.*

PROOF. The order of G is given by $|G|=p^{l^2}\Pi_{i=1}^{l}(p^{2i}-1)$. Hence

$$|G|<p^{l^2}\prod_{i=1}^{l}p^{2i}=p^{l^2}p^{\sum_{i=1}^{l}2i}=p^{l^2+l(l+1)}=p^{2l^2+l}.$$ ■

(2.8) *Let p be a prime, $l\geqq 1$ and let $n=p^l$. Then we have:*

(α) *For $q>13$ the inequality $\gamma_n(q-1)^n>n^2(q-1)|\mathrm{Sp}(2l,p)|$ holds.*

(β) *For $q=11,13$ the inequality $\gamma_n q^n>n^2(q-1)|\mathrm{Sp}(2l,p)|$ holds.*

PROOF. For $n=2,3,4,5$ it is easy to check that (α) and (β) hold, noticing that $\gamma_5\geqq 24$

For $n>5$ we have by (2.6) $(q-1)^{n-1}>p^{2l^2+l}$ and by (2.1) we have $\gamma_n>n^2$. By (2.7) we have $|\mathrm{Sp}(2l,p)|<p^{2l^2+l}$. Consequently we have

$$\gamma_n(q-1)^n=\gamma_n(q-1)(q-1)^{n-1}>n^2(q-1)p^{2l^2+l}>n^2(q-1)|\mathrm{Sp}(2l,p)|$$

for all $q\geqq 11$. ■

(2.9) *Let p be a prime, $l\geqq 1$ and let $n=p^l$. Then we have*

$$\gamma_n q^n>n^2(q-1)|\mathrm{Sp}(2l,p)|,\qquad \textit{for all } q\geqq 11.$$

PROOF. (2.9) is an immediate consequence of (2.8). ■

3. Proofs of Theorem A and Theorem B

THEOREM A. *Denote by $b(n,q)$ the maximal possible order of a solvable, completely reducible subgroup of* $\mathrm{GL}(n,\Delta)$. *Then we have $b(n,q) \leqq \gamma_n q^n$, for all $n \geqq 1$.*

PROOF. Let R be a solvable, completely reducible subgroup of $\mathrm{GL}(n,\Delta)$. The proof of the theorem is by induction on n. For $n = 1$ the assertion of the theorem is trivial. Assume now that $n \geqq 2$ and that the assertion of the theorem holds for every $n_1 < n$. We distinguish 3 cases according as R is primitive, imprimitive and reducible respectively.

Case A. $R = M$ is primitive

Clearly we may assume that M is a maximal primitive solvable subgroup of $\mathrm{GL}(n,\Delta)$. We shall use in Case A the notations introduced in Chapter 1. We distinguish 2 cases according as $m = 1$ and $m > 1$ respectively.

Case 1. $m = 1$

We shall show that in this case the assertion of the theorem is a consequence of (2.9).

Let $n = p_1^{l_1} \cdot p_2^{l_2} \cdots p_k^{l_k}$ be a prime factorization of n (where $p_i \neq p_j$, for $i \neq j$, $1 \leqq i, j \leqq k$). Since $m = 1$ we have here, by the definition of m, $K = E_n \cdot \Delta$, and by (1.2) $F = K^* = E_n \cdot \Delta^*$. Moreover we have $V = M$ and by (1.3) $[A:F] = n^2$ (since here $r = n$). By (1.4) we have

$$[M:A] = [V:A] \leqq \prod_{i=1}^{k} |\mathrm{Sp}(2l_j, p_j)|.$$

Hence with the aid of (1.1) we get

$$|M| \leqq [M:A] \cdot [A:F] \cdot |F| \leqq \left(\prod_{j=1}^{k} |\mathrm{Sp}(2l_j, p_j)| \right) . n^2 . (q-1).$$

But since $n^2 = (p_1^{l_1})^2 \cdot (p_2^{l_2})^2 \cdots (p_k^{l_k})^2$ and since $(q-1) \leqq (q-1)^k$ we have

$$\left(\prod_{j=1}^{k} |\mathrm{Sp}(2l_j, p_j)| \right) n^2 (q-1) \leqq \prod_{j=1}^{k} (p_j^{l_j})^2 (q-1) |\mathrm{Sp}(2l_j, p_j)|.$$

So we conclude

$$|M| \leqq \prod_{j=1}^{k} (p_j^{l_j})^2 (q-1)^{l} |\mathrm{Sp}(2l_j, p_j)|. \tag{3.1}$$

Now S_n contains the direct product $\Gamma_{p_1^{l_1}} \times \Gamma_{p_2^{l_2}} \times \cdots \times \Gamma_{p_k^{l_k}}$ (since $n \geqq p_1^{l_1} + \cdots + p_k^{l_k}$) which is a solvable group of order $\Pi_{j=1}^{k} \gamma_{p_j^{l_j}}$. Hence we have

$$\gamma_n q^n \geqq \gamma_n q^{p_1^{l_1} + p_2^{l_2} + \cdots + p_k^{l_k}} \geqq \prod_{j=1}^{k} \gamma_{p_j^{l_j}} q^{(p_j^{l_j})}.$$

So we conclude

$$\gamma_n q^n \geqq \prod_{j=1}^{k} \gamma_{p_j^{l_j}} q^{(p_j^{l_j})}. \tag{3.2}$$

It is clear now by (3.1) and (3.2) that case 1 is an immediate consequence of (2.9).

Case 2. $m > 1$

Since $r = n/m$ we have here $r < n$. By (1.5) we have $V \subseteq \mathrm{GL}(r, K)$ and V is solvable and irreducible. Hence by induction, since $|K| = q^m$, we have $|V| \leqq \gamma_r (q^m)^r = \gamma_r q^n$. Now $|M| = [M : V]|V|$. By (1.6) we have $[M : V] \leqq m$, consequently $|M| \leqq m\gamma_r q^n$. We shall show now that $\gamma_n \geqq m\gamma_r$. Since $n = m \cdot r$ it follows that S_n contains the direct product

$$\underbrace{\Gamma_r \times \Gamma_r \times \cdots \times \Gamma_r}_{m \text{ times}}$$

which is a solvable group of order γ_r^m. Hence we have $\gamma_n \geqq \gamma_r^m$. If $r > 1$ then in order to show the inequality $\gamma_n \geqq m\gamma_r$ it suffices to show the inequality $\gamma_r^m \geqq m\gamma_r$, i.e. $\gamma_r^{m-1} \geqq m$. Since $r > 1$ we have $\gamma_r \geqq 2$, hence clearly $\gamma_r^{m-1} \geqq m$. If $r = 1$ then $m = n$ and the inequality $\gamma_n \geqq m\gamma_r$ takes the form $\gamma_n \geqq n$ which is trivially true. Consequently we have $\gamma_n \geqq m\gamma_r$ and we conclude $\gamma_n q^n \geqq m\gamma_r q^n \geqq |R|$ as asserted.

Case B. R is imprimitive

By the definition of imprimitivity of a linear group and by the definition of the wreath product of a linear group and a permutation group we clearly may assume the following; There exists a factorization $n = n_1 \cdot n_2$ with $n_1 < n$. There exist a solvable, irreducible subgroup $H_1 \subseteq \mathrm{GL}(n_1, \Delta)$ and a (transitive) solvable subgroup $\Gamma \subseteq S_{n_2}$, such that R is contained in $H_1 \int \Gamma$. By induction we have $|H_1| \leqq \gamma_{n_1} q^{n_1}$. Hence by the definition of a wreath product we have

$$|R| \leqq |H_1 \int \Gamma| = |H_1|^{n_2} |\Gamma| \leqq \gamma_{n_1}^{n_2} \cdot q^{n_1 \cdot n_2} \cdot |\Gamma| = \gamma_{n_1}^{n_2} |\Gamma| \cdot q^n.$$

We shall show that $\gamma_n \geqq \gamma_{n_1}^{n_2} \cdot |\Gamma|$. Clearly S_n contains the group $\Gamma_{n_1} \int \Gamma$ which is solvable of order $\gamma_{n_1}^{n_2} \cdot |\Gamma|$. Hence $\gamma_n \geqq \gamma_{n_1}^{n_2} \cdot |\Gamma|$. So we conclude $\gamma_n q^n \geqq |R|$.

Case C. R is reducible

Since R is completely reducible it follows by the definition of complete reduciblitiy that there exist an integer $t > 1$ and integers $n_i \geqq 1$, $1 \leqq i \leqq t$ with

$\Sigma_{i=1}^t n_i = n$, such that R is conjugate in $\mathrm{GL}(n,\Delta)$ to the group $H_1 \times H_2 \times \cdots \times H_t$. Here H_i is a solvable, irreducible subgroup of $\mathrm{GL}(n_i,\Delta)$, $1 \leqq i \leqq t$. Hence by induction we have

$$|R| = \prod_{i=1}^t |H_i| \leqq \prod_{i=1}^t \gamma_{n_i} q^{n_i} = \left(\prod_{i=1}^t \gamma_{n_i}\right) q^n.$$

But clearly S_n contains the direct product $\Gamma_{n_1} \times \Gamma_{n_2} \times \cdots \times \Gamma_{n_t}$, hence $\gamma_n \geqq \Pi_{i=1}^t \gamma_{n_i}$. Consequently $\gamma_n q^n \geqq |R|$. ■

PROPOSITION 3.1. *Let $R \subseteq \mathrm{GL}(n,\Delta)$ be a solvable, completely reducible subgroup of order $b(n,q)$, where $q > 13$. If $n > 2$ then R is not primitive.*

PROOF. Note that $\mathrm{GL}(n,\Delta)$ contains the subgroup $\Delta^* \int \Gamma_n$ which is by definition solvable and completely reducible of order $\gamma_n(q-1)^n$. Hence $b(n,q) \geqq \gamma_n(q-1)^n$. We shall show that if R is primitive then we must have $|R| < \gamma_n(q-1)^n$. Assume now that $R = M$ is primitive. We distinguish 3 cases according as $m = 1$, $1 < m < n$ and $m = n$ respectively. (Again we use the notation introduced in Section 1.)

Case 1. $m = 1$

In this case we derive that $|R| < \gamma_n(q-1)^n$ exactly in the same way as in the proof of subcase 1 of case A in the proof of Theorem A, using (2.8(α)).

Case 2. $2 \leqq m < n$

Like in the proof of subcase 2 of case A, in the proof of Theorem A we have $|R| \leqq m\gamma_r q^n$. Hence we must show that $\gamma_n(q-1)^n > m\gamma_r q^n$. Now clearly S_n contains the wreath product $\Gamma_r \int \Gamma_m$ which is a solvable group of order $\gamma_r^m \gamma_m$. Hence $\gamma_n \geqq \gamma_r^m \gamma_m$ and it suffices to show that $\gamma_r^m \gamma_m (q-1)^n > m\gamma_r q^n$. But it is easy to see that $\gamma_m \geqq m$ for all m. Hence it suffices to show that $\gamma_r^m (q-1)^n > \gamma_r q^n$. This last inequality is equivalent to the inequality

$$\gamma_r^{m-1} > \left(\frac{q}{q-1}\right)^n.$$

Now since $2 \leqq m < n$ we have $2 \leqq r \leqq n/2$. Hence by (2.2) we have

$$\gamma_r^{m-1} > (1.4)^{r(m-1)} > (1.1)^{2r(m-1)} = (1.1)^{2n-2r} \geqq (1.1)^n.$$

But since $q > 13$ we have $q/(q-1) \leqq 1.1$. Hence we have

$$\gamma_r^{m-1} > (1.1)^n \geqq \left(\frac{q}{q-1}\right)^n.$$

Case 3. $m = n$

Like in the proof of subcase 2 of case A, in the proof of Theorem A we have $|R| \leqq nq^n$. Hence by (2.3) we have $\gamma_n(q-1)^n > nq^n \geqq |R|$. ■

PROPOSITION 3.2. *Let $n > 1$. Assume that Γ_n is transitive. Let $R_1, R_2 \subset \mathrm{GL}(2n, \Delta)$ be subgroups defined by: $R_1 = G_{2,2} \int \Gamma_n$ and $R_2 = \Delta^* \int \Gamma_{2n}$. Then for $|\Delta| = q > 13$ we have $|R_1| < |R_2|$.*

PROOF. Recall that $G_{2,2}$ is defined in (1.7). Clearly $|R_1| = [2(q^2-1)]^n \cdot \gamma_n$ and $|R_2| = (q-1)^{2n}\gamma_{2n}$. Hence we must show $(q-1)^{2n}\gamma_{2n} > [2(q^2-1)]^n\gamma_n$. The last inequality is equivalent to the inequality

$$\frac{\gamma_{2n}}{\gamma_n} > [2(q+1)/(q-1)]^n.$$

But since $q > 13$ we have $q \geqq 16$ and it is easy to check that for $q \geqq 16$ we have $2(q+1)/(q-1) < 12^{1/3}$. Hence by (2.4) we have

$$\frac{\gamma_{2n}}{\gamma_n} \geqq 12^{n/3} > [2(q+1)/(q-1)]^n.$$ ■

PROPOSITION 3.3. *Let $R \subset \mathrm{GL}(2, \Delta)$ be a solvable, completely reducible subgroup of order $b(2, q)$, where $|\Delta| = q \geqq 13$. Then R is conjugate in $\mathrm{GL}(2, \Delta)$ to the group $G_{2,2}$.*

PROOF. If R is reducible, then clearly $R \leqq (q-1)^2$. Hence the assertion of Proposition 3.3 is an easy consequence of (1.7). ■

PROPOSITION 3.4. *Let $R \subset \mathrm{GL}(n, \Delta)$ be a solvable, completely reducible subgroup of order $b(n, q)$, where $q > 13$ and $n > 2$. If R is irreducible then R is conjugate in $\mathrm{GL}(n, \Delta)$ to $\Delta^* \int \Gamma_n$.*

PROOF. By Proposition 3.1 R is not primitive. Hence clearly there exist a factorization $n = l \cdot k$ such that $R = M \int \Gamma$ where M is a primitive solvable subgroup of $\mathrm{GL}(l, \Delta)$ of order $b(l, q)$, and Γ is a solvable subgroup of maximal order in S_k. By Proposition 3.1 we must have either $l = 1$ or $l = 2$. Hence we must have either $n = 2 \cdot k$ and $R = H \int \Gamma$ where $H \subseteq \mathrm{GL}(2, \Delta)$ is a solvable completely reducible subgroup of order $b(2, q)$ and Γ is a solvable subgroup of S_k of order γ_k, or $R = \Delta^* \int \Gamma'$ where Γ' is a solvable subgroup of S_n of order γ_n. By Proposition 3.3 and by (1.8) we clearly have that R is conjugate in $\mathrm{GL}(n, \Delta)$ to $G_{2,2} \int \Gamma_k$ $(n = 2k)$ or R is conjugate in $\mathrm{GL}(n, \Delta)$ to $\Delta^* \int \Gamma_n$. Now if $n = 2k$ and $R \sim G_{2,2} \int \Gamma_k$ then clearly since R is irreducible Γ_k must be a transitive subgroup of S_k. But then by Proposition 3.2 we have $|\Delta^* \int \Gamma_n| > |G_{2,2} \int \Gamma_k|$ $(k > 1$ since

$n > 2$) which is a contradiction to the definition of R. Hence we must have $R \sim \Delta^* \int \Gamma_n$ as asserted. ■

THEOREM B.[†] *Let $R \subset \mathrm{GL}(n, \Delta)$ be a solvable, completely reducible subgroup of order $b(n, q)$, where $|\Delta| = q > 13$ and $n \geqq 1$. Then the following assertion holds*:

(α) *If $n \not\equiv 2 (\mathrm{mod}\ 4)$ or $n \equiv 6 (\mathrm{mod}\ 16)$ then we have $R \sim \Delta^* \int \Gamma_n$.*

(β) *If $n \equiv 2 (\mathrm{mod}\ 4)$ but $n \not\equiv 6 (\mathrm{mod}\ 16)$ then we have $R \sim G_{2,2} \times (\Delta^* \int \Gamma_{n-2})$.*

PROOF. If R is irreducible then if $n = 2$ Theorem B is a consequence of Proposition 3.3. If $n > 2$ then by Proposition 3.4 $R \sim \Delta^* \int \Gamma_n$. But since R is irreducible Γ_n must be transitive, hence by (1.8) $n = 3, 6, 9$ or $n = 4k$ where $k \geqq 1$. Hence clearly the assertion of Theorem B holds in this case.

If R is reducible, then clearly R is conjugate in $\mathrm{GL}(n, \Delta)$ to the group $H_1 \times H_2 \times \cdots \times H_t$, where $H_i \subseteq \mathrm{GL}(n_i, \Delta)$ is a solvable irreducible subgroup of order $b(n_i, q)$, for $1 \leqq i \leqq t$, and $n = \sum_{i=1}^t n_i$ where $n_i \geqq 1$ for $1 \leqq i \leqq t$. Now by Proposition 3.1 either $n_i = 2$ and H_i is conjugate in $\mathrm{GL}(n_i, \Delta)$ to $G_{2,2}$, or H_i is conjugate in $\mathrm{GL}(n_i, \Delta)$ to $\Delta^* \int \Gamma_{n_i}$. If there were $1 \leqq i, j \leqq t$ with $i \neq j$ such that $H_i \sim G_{2,2}$ and $H_j \sim G_{2,2}$ then clearly we can assume without loss of generality that $i = 1$ and $j = 2$. But then we could replace $H_1 \times H_2$ by $\Delta^* \int \Gamma_4$ in the group $H_1 \times H_2 \times \cdots \times H_t$ to obtain a solvable, completely reducible subgroup of order larger than R (since $|G_{2,2}|^2 < (q-1)^4 \cdot 24$). Hence we reach a contradiction. By the above remark we clearly may assume that either H_1 is conjugate in $\mathrm{GL}(2, \Delta)$ to $G_{2,2}$ and H_i is conjugate in $\mathrm{GL}(n_i, \Delta)$ to $\Delta^* \int \Gamma_{n_i}$ for $2 \leqq i \leqq t$, or H_i is conjugate in $\mathrm{GL}(n_i, \Delta)$ to $\Delta^* \int \Gamma_{n_i}$ for all $1 \leqq i \leqq t$. In the first case $H_2 \times \cdots \times H_t$ is conjugate in $\mathrm{GL}(n_2 + n_3 + \cdots + n_t, \Delta)$ to $(\Delta^* \int \Gamma_{n_2}) \times (\Delta^* \int \Gamma_{n_3}) \times \cdots \times (\Delta^* \int \Gamma_{n_t})$ which is clearly conjugate in $\mathrm{GL}(n_2 + n_3 + \cdots + n_t, \Delta)$ to $\Delta^* \int (\Gamma_{n_2} \times \Gamma_{n_3} \times \cdots \times \Gamma_{n_t})$. By the definition of R we clearly must have $\Gamma_{n_2} \times \Gamma_{n_3} \times \cdots \times \Gamma_{n_t}$ is conjugate in $S_{n_2+n_3+\cdots+n_t}$ to $\Gamma_{n_2+n_3+\cdots+n_t}$. Similarly in the second case R is conjugate in $\mathrm{GL}(n, \Delta)$ to $\Delta^* \int \Gamma_n$. Hence we conclude that either $R \sim G_{2,2} \times (\Delta^* \int \Gamma_{n-2})$ or $R \sim \Delta^* \int \Gamma_n$.

If $n \not\equiv 2 (\mathrm{mod}\ 4)$ or $n \equiv 6 (\mathrm{mod}\ 16)$ then by (2.5) we have:

$$|\Delta^* \int \Gamma_n| = (q-1)^n \gamma_n \geqq (q-1)^n \cdot 2\tfrac{2}{3} \gamma_{n-2} = 2\tfrac{2}{3}(q-1)^2 (q-1)^{n-2} \gamma_{n-2}.$$

and $|G_{2,2} \times (\Delta^* \int \Gamma_{n-2})| = |G_{2,2}|(q-1)^{n-2} \gamma_{n-2}$. Since it is easy to check that for $q > 13$ we have $2\tfrac{2}{3}(q-1)^2 > |G_{2,2}|$ we conclude:

(α) If $n \not\equiv 2 (\mathrm{mod}\ 4)$ or $n \equiv 6 (\mathrm{mod}\ 16)$ then $R \sim \Delta^* \int \Gamma_n$.

[†] The assertion of Theorem B fails to be true for $q \leqq 13$ since for $q \leqq 13$, $|G_{2,2} \int S_3| > |\Delta^* \int \Gamma_6|$.

Now if $n \equiv 2 (\mathrm{mod}\ 4)$ but $n \not\equiv 6 (\mathrm{mod}\ 16)$, then by (1.8) $\Gamma_n \sim \Gamma_{n-2} \times S_2$. Hence clearly $\Delta^* \int \Gamma_n \sim (\Delta^* \int \Gamma_{n-2}) \times (\Delta^* \int S_2)$. But

$$|G_{2,2} \times (\Delta^* \int \Gamma_{n-2})| = |\Delta^* \int \Gamma_{n-2}| \cdot |G_{2,2}|$$

and it is easy to check that for $q > 13$ $|G_{2,2}| > |\Delta^* \int S_2|$ and we conclude:

(β) If $n = 2 (\mathrm{mod}\ 4)$ but $n \not\equiv 6 (\mathrm{mod}\ 16)$ then $R \sim G_{2,2} \times (\Delta^* \int \Gamma_{n-2})$. ∎

4. Added remarks

The proofs for all the propositions below can be found in [3].

(4.1) $$b(n, q) \leqq q^{\alpha(q) \cdot n}/24^{1/3} \qquad \text{for all } n \geqq 1$$

where $\alpha(q)$ is defined as follows:

(i) For $q = 8$ or $q \geqq 11$

$$\alpha(q) = (3 \log(q-1) + \log 24)/3 \log q.$$

(ii) For $q = 3, 5, 7, 9$

$$\alpha(q) = (3 \log(q-1) + 4 \log 24)/6 \log q.$$

(iii) $$\alpha(4) = 2/3 + \tfrac{13}{18} \log_2 3.$$

(iv) $$\alpha(2) = 1 + \tfrac{2}{3} \log_2 3.$$

Note that (4.1)(i) is a slight improvement of (0.1) of the Introduction.

We mention that it is easy to see that $\alpha(q)$ is a monotonic decreasing function of q for $q \geqq 11$, and $\lim_{q \to \infty} \alpha(q) = 1$.

The following table gives upper bounds for $\alpha(q)$ where $q = 2, 3, 4, 5, 7, 8, 9, 11$; the data are accurate up to two decimal digits.

q	2	3	4	5	7	8	9	11
$\alpha(q)$	2.056	2.244	1.812	1.747	1.5491	1.446	1.438	1.402

(4.2) The bounds given for $b(n, q)$ in (4.1) are the best in the following sense: If $b(n, q) \leqq D \cdot q^{\delta(q) \cdot n}$ for all $n \geqq 1$, then we have $\delta(q) \geqq \alpha(q)$, and if $\delta(q) = \alpha(q)$ then $D \geqq 1/24^{1/3}$.

(4.3) Let $R \subseteq \mathrm{GL}(n, \Delta)$ be a solvable, completely reducible subgroup of order $b(n, 3)$ (here $|\Delta| = 3$). Then we have:

(i) If $n = 2k$ is even, then $R \sim \mathrm{GL}(2, \Delta) \int \Gamma_k$.

(ii) If $n = 2k + 1$ is odd, then $R \sim (\mathrm{GL}(2, \Delta) \int \Gamma_k) \times \Delta^*$.

We mention that for $|\Delta| = 11$ the solvable, completely reducible subgroups of $\mathrm{GL}(n, \Delta)$ of order $b(n, 11)$ do not form a single conjugacy class. Indeed the groups $G_{2,2}$ and $G_{2,3}$ which are contained in $\mathrm{GL}(2, \Delta)$ and are introduced in (1.7) have the same order, 240, and are not conjugate in $\mathrm{GL}(2, \Delta)$. (It is easy to see that $b(2, 11) = 240$.) For $|\Delta| = 7$, $\mathrm{GL}(3, \Delta)$ possesses at least two conjugacy classes of solvable, completely reducible subgroups of order $b(3, 7)$ which are the following: the primitive solvable subgroups of order 6^4 (see [4], thm. 6, pp. 167) and the subgroups conjugate to $\Delta^* \int S_3$ whose order is also 6^4. For $|\Delta| = 2, 4, 5, 8, 9, 13$ we do not know whether the solvable, completely reducible subgroups of order $b(n, q)$ form a unique conjugacy class.

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